Last Time: Limits

Recall Curves Criterion: -A function f of several variables satisfies $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L$ if and only if for all continuous space curves $\vec{r}(t)$ with $\lim_{t\to\infty} \vec{r}(t) = \vec{a}$ and $\vec{r}(t) \neq R$ for all t we have $\lim_{t\to\infty} f(\vec{r}(t)) = L$.

-We saw an example showing $\lim_{\vec{x}\to\vec{a}} f(\vec{x})$ DNE...

Idea: Find two curves $\vec{r}_0(t)$ and $\vec{r}_1(t)$ with $\lim_{t\to\infty} \vec{r}_i(t)$ and $\lim_{t\to\infty} f(\vec{r}_0(t)) \neq \lim_{t\to\infty} f(\vec{r}_1(t))$

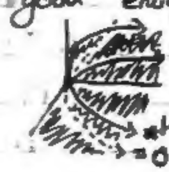
NB: The collection of lines

$l_{a,b}(t) = \vec{a} + t\langle a,b\rangle$

is a great collection of curves to work with to solve b/c $\lim_{t\to 0} l_{a,b}(t) = \vec{a}$

NB: These are not always good enough

Ex: Let $f(x,y) = \begin{cases} 1 & \text{if } y = x^2 \\ 0 & \text{otherwise} \end{cases}$



Claim: $\lim_{\vec{x}\to\vec{0}} f(\vec{x})$ DNE

-For lines $l_{a,b}(t)$:

$\lim_{t\to 0^+} f(l_{a,b}(t)) = \lim_{t\to 0^+} f(at, bt)$

off the origin, (at, bt) satisfies $bt = (at)^2$ at

IE: $f(at,bt) = 0$ for all but finitely many times

IE: $\lim_{t\to 0^+} f(l_{a,b}(t)) = 0$

-On the other hand, letting $\vec{r}(t) = \langle t, t^2\rangle$, we see

$f(\vec{r}(t)) = f(t, t^2) = 1$ for all t.

IE: $\lim_{t\to 0} f(\vec{r}(t)) = \lim_{t\to 0} 1 = 1$

-Thus since $0 \neq 1$, $\lim_{\vec{x}\to\vec{0}} f(\vec{x})$ DNE!

Q: How can we show when limits do exist?

A Trick: Try polar coordinates...

Ex: Does $\lim_{(x,y)\to\vec{0}} \frac{\sin(x^2+y^2)}{x^2+y^2}$ exist?

$\longrightarrow$

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$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ (x,y)\to(0,0) \Rightarrow r\to 0^+ \end{cases}$$

Sol: Lets convert to polar coords!

$$\lim_{(x,y)\to\vec{0}} \frac{\sin(x^2+y^2)}{x^2+y^2} \overset{\text{if it exists}}{=} \lim_{r\to 0^+} \frac{\sin((r\cos\theta)^2+(r\sin\theta)^2)}{(r\cos\theta)^2+(r\sin\theta)^2}$$

$$= \lim_{r\to 0^+} \frac{\sin(r^2\cancel{(\cos^2\theta+\sin^2\theta)}^{1})}{r^2\cancel{(\cos^2\theta+\sin^2\theta)}^{1}}$$

$$= \lim_{r\to 0^+} \frac{\sin(r^2)}{r^2} \longrightarrow \frac{0}{0}\text{-type}$$

$$\boxed{LH} = \lim_{r\to 0^+} \frac{\cancel{2r}\cos(r^2)}{\cancel{2r}} = \lim_{r\to 0^+} \cos(r^2) = \cos(0^2) = 1$$

Ex: Does $\lim_{(x,y)\to\vec{0}} \frac{x^2-y^2}{x^2+y^2}$ exist?

Sol: $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{r\to 0^+} \frac{(r\cos\theta)^2-(r\sin\theta)^2}{(r\cos\theta)^2+(r\sin\theta)^2}$$

$$= \lim_{r\to 0^+} \frac{\cancel{r^2}(\cos^2\theta-\sin^2\theta)}{\cancel{r^2}}$$

$$= \lim_{r\to 0^+} \cos^2\theta - \sin^2\theta$$

$$= \lim_{r\to 0^+} \cos(2\theta) = \cos(2\theta)$$

Dependent on θ

If we approach along an angle of $\theta = \pi/2$, we expect $\lim_{\vec{x}\to\vec{0}} f(\vec{x}) = \cos(2\cdot\pi/2) = -1$. Where as approaching at $\theta = 0$ yields $\lim_{\vec{x}\to\vec{0}} f(\vec{x}) = \cos(2\cdot 0) = 1$

$\therefore$ $\lim_{r\to 0^+} \frac{x^2-y^2}{x^2+y^2}$ DNE!

Def: A function f of m-variables is continuous at $\vec{a} \in dom(f)$ when $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$

f is continuous on set D when f is cts at every value of D

Ex: Every polynomial in n-variables is cts on $\mathbb{R}^n$

Ex: Every rational function of n-variables is cts on its domain

E.g. $\frac{x^2-y^2}{x^2+y^2}$ is cts on its domain

i.e. it is cts everywhere but $(0,0)$.

Ex: $\frac{\sin(x^2+y^2)}{x^2+y^2}$ is cts everywhere but $(0,0)$.

OTOH: $f(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$ is cts everywhere (why?)

NB: Usual "rules" for continuity apply. (Calc I)

$\longrightarrow$

Derivatives of Multivariable Functions

Idea: The derivative measures change in output from corresponding small changes in input...

IN SOME DIRECTION

Def: Let f be a function of n-variables and $\vec{u}$ a unit vector in $\mathbb{R}^n$. Let $\vec{a} \in \text{dom}(f)$.

- The directional derivative of f at $\vec{a}$ in direction of $\vec{u}$ is $D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0^+} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h}$

Ex: Compute the directional derivative of $f(x,y) = xy$ at $\vec{a} = \langle 1,3 \rangle$ in direction $\vec{u} = \frac{1}{2}\langle \sqrt{2}, \sqrt{2} \rangle$

Sol: $D_{\vec{u}} f(\vec{a}) = \lim_{h \to 0^+} \frac{f(\vec{a}+h\vec{u}) - f(\vec{a})}{h}$

$$= \lim_{h \to 0^+} \frac{f(1+\frac{\sqrt{2}}{2}h, 3+\frac{\sqrt{2}}{2}h) - f(1,3)}{h}$$

$$= \lim_{h \to 0^+} \frac{(1+\frac{\sqrt{2}}{2}h)(3+\frac{\sqrt{2}}{2}h) - 1 \cdot 3}{h}$$

$$= \lim_{h \to 0^+} \frac{\cancel{3} + h(\frac{3\sqrt{2}}{2} + \frac{\sqrt{2}}{2}) + h^2 - \cancel{3}}{h}$$

$$= \lim_{h \to 0^+} \frac{h(2\sqrt{2}+h)}{h} = \lim_{h \to 0^+} (2\sqrt{2}+h) = 2\sqrt{2} + 0 = \boxed{2\sqrt{2}}$$

Exercise: Repeat example with $\vec{a} = \langle x,y \rangle$

NB: The directional derivative is very general.

We want something like the "rules" from Calc I...

Def: Let f be a function of n-variables and let $\vec{e}_k$ be "k-th standard basic vector in $\mathbb{R}^n$"

i.e. $\vec{e}_k = \langle 0, 0, \ldots, 1, \ldots, 0 \rangle$ k^{th} position

- The k^{th} partial derivative of f (alt. the partial derivative of f wrt x_k) is $D_{\vec{e}_k} f(\vec{a})$